GEODYNAMICS

Applications of Continuum Physics to Geological Problems

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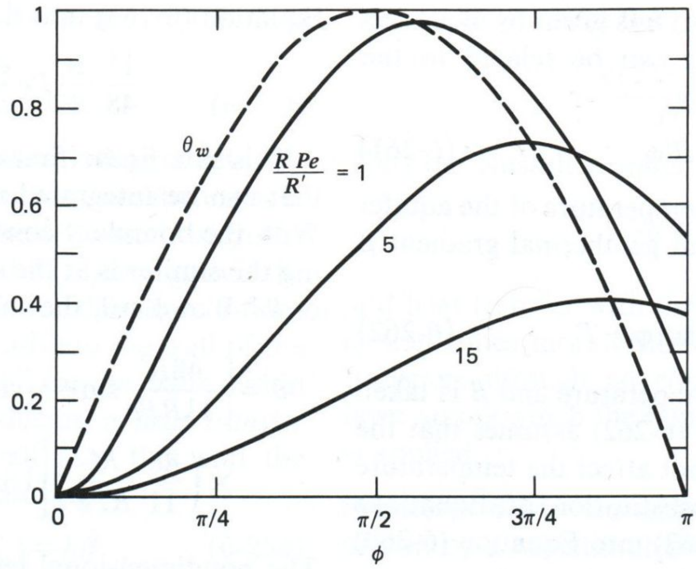

Figure 6-35 Dimensionless mean water temperature in the aquifer as a function of position for three nondimensional flow rates. The dashed line is the dimensionless aquifer wall temperature.

$\theta_e = 1/2$ corresponds to $\overline{T}_e = T_0 + \frac{1}{2}\beta R'$, and T_w at $\phi = \pi/2$ is $T_0 + \beta R'$ ($T_0 \ll \beta R'$).

In order to better understand why there is a maximum exit temperature, we show the mean temperature of the water in the aquifer as a function of position in Figure 6-35 for three flow rates. The dimensionless wall or rock temperature θ_w,

$$\theta_w = \frac{T_w - T_0}{\beta R'} \tag{6-271}$$

is also given in the figure. For a low flow rate, $RPe/R' = 1$, for example, the water temperature follows the wall temperature because of the large heat transfer, and the exit temperature is low. For very slow flow, $RPe/R' \to 0$, the water temperature equals the wall temperature $\theta = \theta_w = \sin\phi$, the exit temperature equals the entrance temperature, and there is no hot spring. For a high flow rate, $RPe/R' = 15$, for example, there is very little heat transfer, and the water does not heat up. In the limit $RPe/R' \to \infty$ the water temperature everywhere in the aquifer equals the entrance temperature, and there is no hot spring. The case of maximum exit temperature, $RPe/R' = 5$ and $\theta_e = 0.52$ is also shown in Figure 6-35.

Although the analysis given here has been greatly simplified, the results are applicable to the more general problem in which the temperature distribution in the rock through which the aquifer passes must also be determined. This requires a solution of Laplace's equation. Also, the transition to turbulence must be considered. The more complete solutions require numerical methods. However, the results show that the maximum temperature that can be expected from a hot spring is about one-half the temperature obtained by extrapolating the regional geothermal gradient to the base of the aquifer, similar to the result obtained here.

Problem 6-25 Verify by direct substitution that Equation (6-269) is the solution of Equation (6-268).

Problem 6-26 The results of this section were based on the assumption of a laminar heat transfer coefficient for the aquifer flow. Since this requires $Re < 2200$, what limitation is placed on the Péclet number?

6-17 THERMAL CONVECTION

As discussed in Section 1-13, plate tectonics is a consequence of thermal convection in the mantle driven largely by radiogenic heat sources and the cooling of the earth. When a fluid is heated, its density generally decreases because of thermal expansion. A fluid layer that is heated from below or from within and cooled from above has dense cool fluid near the upper boundary and hot light fluid at depth. This situation is gravitationally unstable, and the cool fluid tends to sink and the hot fluid tends to rise. This is thermal convection. The phenomenon is illustrated in Figure 1-59.

Appropriate forms of the continuity, force balance, and temperature equations for two-dimensional flow are required for a quantitative study of thermal convection. Density variations caused by thermal expansion lead to the buoyancy forces that drive thermal convection. Thus it is essential to account for density variations in the gravitational body force term of the conservation of momentum or force balance equation. In all other respects, however, the density variations are sufficiently small so that they can be neglected. This is known as the *Boussinesq approximation*. It allows us to use the incompressible conservation of fluid equation (6-53). The force balance equations (6-64) and (6-65) are also applicable. However, to account for the buoyancy forces, we must allow for small density variations in the vertical force balance, Equation (6-65), by letting

$$\rho=\rho_0+\rho' \tag{6-272}$$

where ρ_0 is a reference density and $\rho' \ll \rho_0$. Equation (6-65) can then be written

$$0=-\frac{\partial p}{\partial y}+\rho_0 g+\rho' g+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{6-273}$$

We can eliminate the hydrostatic pressure corresponding to the reference density by introducing

$$P=p-\rho_0 g y \tag{6-274}$$

as in Equation (6-66). The horizontal and vertical equations of motion, Equations (6-64) and (6-273), become

$$0=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{6-275}$$

$$0=-\frac{\partial P}{\partial y}+\rho' g+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{6-276}$$

Density variations caused by temperature changes are given by Equation (4-172)

$$\rho'=-\rho_0\alpha_v(T-T_0) \tag{6-277}$$

where α_v is the volumetric coefficient of thermal expansion and T_0 is the reference temperature corresponding to the reference density ρ_0. Substitution of Equation (6-277) into Equation (6-276) gives

$$0=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-g\rho_0\alpha_v(T-T_0) \tag{6-278}$$

The last term in Equation (6-278) is the buoyancy force per unit volume. The gravitational buoyancy term depends on temperature. Thus the velocity field cannot be determined without simultaneously solving for the temperature field. Therefore we require the heat equation that governs the variation of temperature.

The energy balance must take account of heat transport by both conduction and convection. Consider the small two-dimensional element shown in Figure 6-36. Since the thermal energy content of the fluid is ρcT per unit volume, an amount of heat $\rho cTu\,\delta y$ is transported across the right side of the element by the velocity component u in the x direction. This is an energy flow per unit time and per unit depth or distance in the dimension perpendicular to the figure. If ρcTu is the energy flux at x, then $\rho cTu+\partial/\partial x(\rho cTu)\,\delta x$ is the energy flow rate per unit area at $x+\delta x$. The net

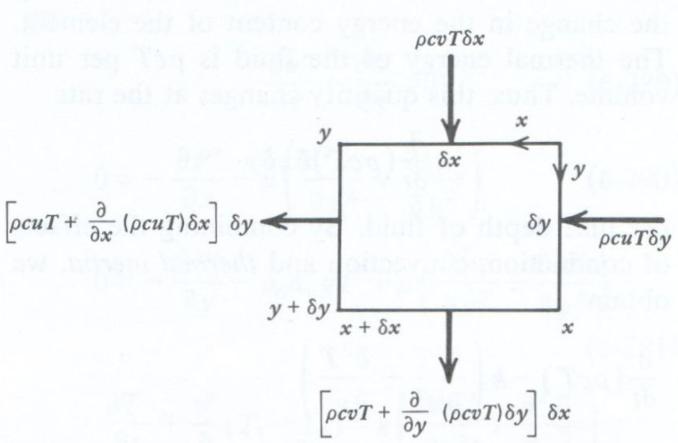

Figure 6-36 Heat transport across the surfaces of an infinitesimal rectangular element by convection.

energy advected out of the elemental volume per unit time and per unit depth due to flow in the x direction is thus

$$\left[\left\{\rho cTu+\frac{\partial}{\partial x}(\rho cTu)\delta x\right\}-\rho cTu\right]\delta y = \frac{\partial}{\partial x}(\rho cTu)\delta x\delta y \tag{6-279}$$

The same analysis applied in the y direction gives

$$\left[\left(\rho cTv+\frac{\partial}{\partial y}\{\rho cTv\}\delta y\right)-\rho cTv\right]\delta x = \frac{\partial}{\partial y}(\rho cTv)\delta x\delta y \tag{6-280}$$

for the net rate at which heat is advected out of the element by flow in the y direction per unit depth. Thus, the net rate of heat advection out of the element by flow in both directions is

$$\left[\frac{\partial}{\partial x}(\rho cTu)+\frac{\partial}{\partial y}(\rho cTv)\right]\delta x\delta y$$

per unit depth. We have already derived the expression for the net rate at which heat is conducted out of the element, per unit depth, in Equation (4-49); it is

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)\delta x\delta y$$

Conservation of energy states that the combined transport of energy out of the elemental volume by conduction and convection must be balanced by the change in the energy content of the element. The thermal energy of the fluid is ρcT per unit volume. Thus, this quantity changes at the rate

$$\frac{\partial}{\partial t}(\rho cT)\delta x\delta y$$

per unit depth of fluid. By combining the effects of conduction, convection and *thermal inertia*, we obtain

$$\frac{\partial}{\partial t}(\rho cT)-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) + \frac{\partial}{\partial x}(\rho cuT)+\frac{\partial}{\partial y}(\rho cvT)=0 \tag{6-281}$$

By treating ρ and c as constants and noting that

$$\frac{\partial}{\partial x}(uT)+\frac{\partial}{\partial y}(vT)=u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+T\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} \tag{6-282}$$

(the last step following as a consequence of the continuity equation) and $\kappa=k/\rho c$, we finally arrive at the heat equation for two-dimensional flows

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{6-283}$$

In deriving Equation (6-283), we have neglected some factors which contribute to a general energy balance but which are negligible in our present application. These include frictional heating in the fluid associated with the resistance to flow and compressional heating associated with the work done by pressure forces in moving the fluid. We have already derived and used simplified forms of this equation in Section 4-19.

6-18 LINEAR STABILITY ANALYSIS FOR THE ONSET OF THERMAL CONVECTION IN A LAYER OF FLUID HEATED FROM BELOW

The layer of fluid illustrated in Figure 6-37 is heated from below; that is, its upper surface $y=-b/2$ is maintained at the relatively cold reference temperature T_0 and its lower boundary $y=b/2$ is kept at the relatively hot temperature $T_1(T_1>T_0)$. We assume that there are no heat sources in the fluid. Buoyancy forces tend to drive convection in the fluid layer. Fluid near the heated lower boundary becomes hotter and lighter than the overlying fluid and tends to rise. Similarly, fluid near the colder, upper boundary is denser than the fluid below and tends to sink. However, the motion does not take place for small temperature differences across the layer because the viscous resistance of the medium to flow must be overcome. We use the equations of the preceding section to determine the conditions, for example, the

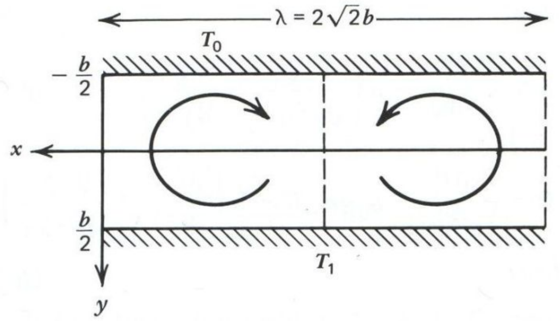

Figure 6-37 Two-dimensional cellular convection in a fluid layer heated from below.

minimum temperature difference, required for convection to occur.

In the absence of convection, that is, for $T_1 - T_0$ sufficiently small, the fluid is stationary ($u = v = 0$), and we can assume that a steady ($\partial/\partial t = 0$) conductive state with $\partial/\partial x = 0$ exists. The energy equation (6-283) then simplifies to

$$\frac{d^2 T_c}{dy^2} = 0 \tag{6-284}$$

where the subscript c indicates that this is the conduction solution. The solution of Equation (6-284) that satisfies the boundary conditions $T = T_0$ at $y = -b/2$ and $T = T_1$ at $y = +b/2$ is the linear temperature profile

$$T_c = \frac{T_1 + T_0}{2} + \frac{(T_1 - T_0)}{b} y \tag{6-285}$$

If one imagines gradually increasing the temperature difference across the layer ($T_1 - T_0$), the stationary conductive state will persist until $T_1 - T_0$ reaches a critical value at which even the slightest further increase in temperature difference will cause the layer to become unstable and convection to occur. Thus, at the *onset of convection* the fluid temperature is nearly the conduction temperature profile and the temperature difference T',

$$T' \equiv T - T_c = T - \frac{(T_1 + T_0)}{2} - \frac{(T_1 - T_0)}{b} y \tag{6-286}$$

is arbitrarily small. The convective velocities u', v' are similarly infinitesimal when motion first takes place.

The form of the energy equation that pertains to the onset of convection can be written in terms of T' by solving Equation (6-286) for T and substituting into Equation (6-283). One gets

$$\frac{\partial T'}{\partial t} + u' \frac{\partial T'}{\partial x} + v' \frac{\partial T'}{\partial y} + \frac{v'(T_1 - T_0)}{b} = \kappa \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} \right) \tag{6-287}$$

Since T', u', v' are small quantities, the *nonlinear terms* $u' \partial T'/\partial x$ and $v' \partial T'/\partial y$ on the left side of Equation (6-287) are much smaller than the remaining linear terms in the equation. Thus they can be neglected and Equation (6-287) can be written as

$$\frac{\partial T'}{\partial t} + \frac{v'}{b}(T_1 - T_0) = \kappa \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} \right) \tag{6-288}$$

The neglect of the nonlinear terms, the terms involving products of the small quantities u', v', and T', is a standard mathematical approach to problems of stability. Our analysis for the conditions in the fluid layer at the onset of convection is known as a *linearized stability analysis*. It is a valid approach for the study of the onset of convection when the motions and the thermal disturbance are infinitesimal.

To summarize, the equations for the small perturbations of temperature T', velocity u', v', and pressure P' when the fluid layer becomes unstable are

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{6-289}$$

$$0 = -\frac{\partial P'}{\partial x} + \mu \left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} \right) \tag{6-290}$$

$$0 = -\frac{\partial P'}{\partial y} - \rho_0 \alpha_v g T' + \mu \left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} \right) \tag{6-291}$$

$$\frac{\partial T'}{\partial t} + \frac{v'}{b}(T_1 - T_0) = \kappa \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} \right) \tag{6-292}$$

From the second term on the right side of the vertical force balance equation (6-291), it is seen that we have taken the buoyancy force at any point in the layer to depend only on the departure of the fluid temperature from the basic conduction temperature at the point. The conduction temperature profile of the stationary state is the reference temperature profile against which buoyancy forces are determined.

Equations (6-289) to (6-292) are solved subject to the following boundary conditions. We assume that the surfaces $y=\pm b/2$ are isothermal and that no flow occurs across them; that is,

$$T'=v'=0 \text{ on } y=\pm\frac{b}{2} \tag{6-293}$$

If the boundaries of the layer are solid surfaces,

$$u'=0 \text{ on } y=\pm\frac{b}{2} \tag{6-294}$$

This is the no-slip condition requiring that there be no relative motion between a viscous fluid and a bounding solid surface at the solid-fluid interface. If the surfaces $y=\pm b/2$ are free surfaces, that is, if there is nothing at $y=\pm b/2$ to exert a shear stress on the fluid, u' need not vanish on the boundaries. Instead, the shear stress τ'_{yx} must be zero on $y=\pm b/2$. From Equation (6-58) this requires

$$\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}=0 \text{ on } y=\pm\frac{b}{2} \tag{6-295}$$

Conditions (6-295) can be simplified even further because $v'=0$ on $y=\pm b/2$ for any x and consequently $\partial v'/\partial x\equiv 0$ on $y=\pm b/2$. The free surface boundary conditions are therefore

$$\frac{\partial u'}{\partial y}=0 \text{ on } y=\pm\frac{b}{2} \tag{6-296}$$

A simple analytic solution can be obtained for the linearized stability problem if the free surface conditions (6-296) are adopted.

We once again introduce the stream function defined in Equations (6-69) and (6-70). Thus the conservation equation (6-289) is automatically satisfied, and Equations (6-290) to (6-292) can be written

$$0=-\frac{\partial P'}{\partial x}-\mu\left(\frac{\partial^3\psi'}{\partial x^2\partial y}+\frac{\partial^3\psi'}{\partial y^3}\right) \tag{6-297}$$

$$0=-\frac{\partial P'}{\partial y}-\rho_0 g\alpha_v T'+\mu\left(\frac{\partial^3\psi'}{\partial x^3}+\frac{\partial^3\psi'}{\partial y^2\partial x}\right) \tag{6-298}$$

$$\frac{\partial T'}{\partial t}+\frac{1}{b}(T_1-T_0)\frac{\partial\psi'}{\partial x}=\kappa\left(\frac{\partial^2 T'}{\partial x^2}+\frac{\partial^2 T'}{\partial y^2}\right) \tag{6-299}$$

Eliminating the pressure from Equations (6-297) and (6-298) yields

$$0=\mu\left(\frac{\partial^4\psi'}{\partial x^4}+2\frac{\partial^4\psi'}{\partial x^2\partial y^2}+\frac{\partial^4\psi'}{\partial y^4}\right)-\rho_0 g\alpha_v\frac{\partial T'}{\partial x} \tag{6-300}$$

The problem has now been reduced to the solution of two simultaneous partial differential equations (6-299) and (6-300) for the two variables ψ' and T'.

Because these equations are linear equations with constant coefficients, we can solve them by the method of separation of variables. The boundary conditions (6-293) and (6-296) are automatically satisfied by solutions of the form

$$\psi'=\psi'_0\cos\frac{(\pi y)}{b}\sin\left(\frac{2\pi x}{\lambda}\right)e^{\alpha' t} \tag{6-301}$$

$$T'=T'_0\cos\frac{(\pi y)}{b}\cos\left(\frac{2\pi x}{\lambda}\right)e^{\alpha' t} \tag{6-302}$$

The velocity and temperature perturbations described by these equations are horizontally periodic disturbances with wavelength λ and maximum amplitudes ψ'_0 and T'_0. The value of α' determines whether or not the disturbances will grow in time. For α' positive, the disturbances will amplify, and the heated layer is convectively unstable. For α' negative, the disturbances will decay in time, and the layer is stable against convection. We can determine α' by substituting Equations (6-301) and (6-302) into Equations (6-299) and

(6-300). We find

$$\left(\alpha' + \frac{\kappa\pi^2}{b^2} + \frac{\kappa 4\pi^2}{\lambda^2}\right) T_0' = -\frac{(T_1 - T_0)2\pi}{\lambda b}\psi_0' \tag{6-303}$$

$$\mu\left(\frac{4\pi^2}{\lambda^2} + \frac{\pi^2}{b^2}\right)^2 \psi_0' = -\frac{2\pi}{\lambda}\rho_0 g \alpha_v T_0' \tag{6-304}$$

The disturbance amplitudes ψ_0' and T_0' can be eliminated from these equations by division, yielding an equation that can be solved for α'. The *growth rate* α' is found to be

$$\alpha' = \frac{\kappa}{b^2}\left\{\left(\frac{\rho_0 g \alpha_v b^3 (T_1 - T_0)}{\mu\kappa}\right)\left(\frac{\dfrac{4\pi^2 b^2}{\lambda^2}}{\left(\dfrac{4\pi^2 b^2}{\lambda^2} + \pi^2\right)^2}\right) - \left(\pi^2 + \frac{4\pi^2 b^2}{\lambda^2}\right)\right\} \tag{6-305}$$

The dimensionless growth rate $\alpha' b^2/\kappa$ is seen to depend on only two quantities, $2\pi b/\lambda$, a dimensionless wave number, and a dimensionless combination of parameters known as the *Rayleigh number Ra*

$$Ra = \frac{\rho_0 g \alpha_v (T_1 - T_0) b^3}{\mu\kappa} \tag{6-306}$$

In terms of the Rayleigh number we can write Equation (6-305) as

$$\frac{\alpha' b^2}{\kappa} = \frac{Ra\dfrac{4\pi^2 b^2}{\lambda^2} - \left(\pi^2 + \dfrac{4\pi^2 b^2}{\lambda^2}\right)^3}{\left(\pi^2 + \dfrac{4\pi^2 b^2}{\lambda^2}\right)^2} \tag{6-307}$$

The growth rate is positive and there is instability if

$$Ra > \frac{\left(\pi^2 + \dfrac{4\pi^2 b^2}{\lambda^2}\right)^3}{\dfrac{4\pi^2 b^2}{\lambda^2}} \tag{6-308}$$

The growth rate is negative and there is stability if Ra is less than the right side of Equation (6-308). Convection just sets in when $\alpha' = 0$, which occurs when

$$Ra \equiv Ra_{cr} = \frac{\left(\pi^2 + \dfrac{4\pi^2 b^2}{\lambda^2}\right)^3}{\dfrac{4\pi^2 b^2}{\lambda^2}} \tag{6-309}$$

The *critical value of the Rayleigh number* Ra_{cr} marks the onset of convection. If $Ra < Ra_{cr}$, disturbances will decay with time; if $Ra > Ra_{cr}$, perturbations will grow exponentially with time.

According to Equation (6-309), the critical Rayleigh number is a function of the wavelength of the disturbance. Figure 6-38 shows how Ra_{cr} depends on $2\pi b/\lambda$. If the Rayleigh number and disturbance wavelength are such that the point lies above the curve, the perturbation of wavelength λ is unstable; if the point lies below the curve, convection cannot occur with disturbances of wavelength λ. For example, if $Ra = 2000$, all disturbances with $0.8 \lesssim 2\pi b/\lambda \lesssim 5.4$ are convectively unstable. However, convection cannot occur for $2\pi b/\lambda \lesssim 0.8$ and $2\pi b/\lambda \gtrsim 5.4$. Figure 6-38 shows that there is a minimum value of Ra_{cr}. If Ra lies below the minimum value all disturbances decay, the layer is stable, and convection cannot occur.

The value of $2\pi b/\lambda$ at which Ra_{cr} is a minimum can be obtained by setting the derivative of the right side of Equation (6-309) with respect to $2\pi b/\lambda$ equal to zero. One obtains

$$\frac{\partial Ra_{cr}}{\partial\left(\dfrac{2\pi b}{\lambda}\right)} = \left[\frac{4\pi^2 b^2}{\lambda^2} 3\left(\pi^2 + \frac{4\pi^2 b^2}{\lambda^2}\right)^2 2\left(\frac{2\pi b}{\lambda}\right) - \left(\pi^2 + \frac{4\pi^2 b^2}{\lambda^2}\right)^3 2\left(\frac{2\pi b}{\lambda}\right)\right]\left(\frac{4\pi^2 b^2}{\lambda^2}\right)^{-2} = 0 \tag{6-310}$$

or

$$\frac{2\pi b}{\lambda} = \frac{\pi}{\sqrt{2}} \tag{6-311}$$

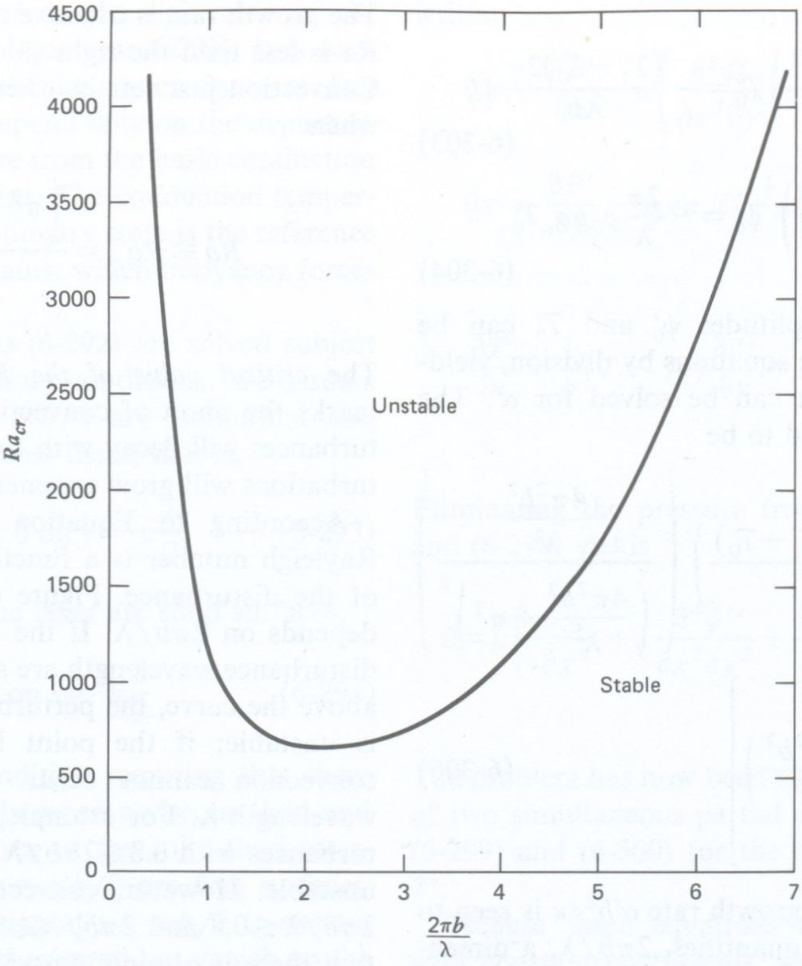

Figure 6-38 Critical Rayleigh number Ra_{cr} for the onset of convection in a layer heated from below with stress-free boundaries as a function of dimensionless wave number $2\pi b/\lambda$.

The value of the wavelength corresponding to the smallest value of the critical Rayleigh number is

$$\lambda = 2\sqrt{2}\, b \qquad (6\text{-}312)$$

Substitution of this value for the wavelength back into Equation (6-309) gives the minimum critical Rayleigh number

$$\min(Ra_{cr}) = \frac{27\pi^4}{4} = 657.5 \qquad (6\text{-}313)$$

The requirement that Ra exceed Ra_{cr} for convection to occur can be restated in a number of more physical ways. One can think of the temperature difference across the layer as having to exceed a certain minimum value or the viscosity of the fluid as having to lie below a critical value before convection sets in. If Ra is increased from 0, for example, by increasing $T_1 - T_0$, other quantities remaining fixed, convection sets in when Ra reaches 657.5 (for heating from below with stress-free boundaries), and the aspect ratio of each convection cell is $\sqrt{2}$, as shown in Figure 6-37. The minimum value of Ra_{cr} and the disturbance wavelength for which Ra_{cr} takes the minimum value must be determined numerically for no-slip velocity boundary conditions. For that case min Ra_{cr} = 1707.8 and $\lambda = 2.016b$.

The linear stability analysis for the onset of convection can also be carried out for a fluid layer heated uniformly from within and cooled from above. The lower boundary is assumed to be insulating; that is, no heat flows across the boundary. Once again the fluid near the upper boundary is cooler and more dense than the fluid beneath.

Therefore buoyancy forces can drive fluid motion provided they are strong enough to overcome the viscous resistance. This type of instability is directly applicable to the earth's mantle, since the interior of the earth is heated by the decay of the radioactive elements and the near-surface rocks are cooled by heat conduction to the surface. These near-surface rocks are cooler and more dense than the hot mantle rocks at depth. The appropriate Rayleigh number for a fluid layer heated from within is

$$Ra = \frac{\alpha_v \rho_0^2 g H b^5}{k \mu \kappa} \tag{6-314}$$

where H is the rate of internal heat generation per unit mass. For no-slip velocity boundary conditions, the minimum critical Rayleigh number is 2772, and the associated value of $2\pi b/\lambda$ is 2.63; for free-slip conditions min $Ra_{cr} = 867.8$, and the associated value of $2\pi b/\lambda$ is 1.79.

We can estimate the value of this Rayleigh number for the mantle of the earth. Based on the postglacial rebound studies we take $\mu = 10^{21}$ Pa s. For the rock properties we take $k = 4$ W m^{-1} °K^{-1}, $\kappa = 1$ mm^2 s^{-1}, and $\alpha_v = 3 \times 10^{-5}$ °K^{-1}. We assume $g = 10$ m s^{-2} and an average density $\rho_0 =$ 4000 kg m^{-3}. Based on our discussion of the distribution of heat sources in the mantle (see Chapter 4) we take $H = 9 \times 10^{-12}$ W kg^{-1}. If convection is restricted to the upper mantle, it is reasonable to take $b = 700$ km. We find that $Ra = 2 \times 10^6$. If we apply the same values to the entire mantle and take $b = 2880$ km, we find that $Ra = 2 \times 10^9$. In either case the calculated value for the Rayleigh number is much greater than the minimum critical value. It was essentially this calculation that led Arthur Holmes to propose in 1931 that thermal convection in the mantle was responsible for driving continental drift.

Problem 6-27 Estimate the values of the Rayleigh numbers for the mantles of Mercury, Venus, Mars, and the Moon. Assume heat is generated internally at the same rate it is produced in the earth. Use the same values for μ, k, κ, and α_v as used above for the earth's mantle. Obtain appropriate values of ρ_0, g, and b from the discussion of Chapter 1.

Problem 6-28 Calculate the exact minimum and maximum values of the wavelength for disturbances that are convectively unstable at $Ra = 2000$. Consider a fluid layer heated from below with free-slip boundary conditions.

Problem 6-29 Formulate the linear stability problem for the onset of convection in a layer of fluid heated from within. Assume that the boundaries are stress-free. Take the upper boundary to be isothermal and the lower boundary to be insulating. Carry the formulation to the point where the solution to the problem depends only on the integration of a single ordinary differential equation for the stream function subject to appropriate boundary conditions.

6-19 BOUNDARY LAYER THEORY FOR FINITE-AMPLITUDE THERMAL CONVECTION

The linear stability theory given in the previous section determines whether or not thermal convection occurs. However, it is not useful in determining the structure of convection when the Rayleigh number exceeds the critical value. Because it is linear, the stability analysis cannot predict the magnitude of *finite-amplitude convective flows*. In order to do this, it is necessary to solve the full nonlinear equations.

For large values of the Rayleigh number a *boundary layer analysis* can be used to determine the structure of the convection cells. Again we consider a fluid layer of thickness b heated from below. The upper boundary is maintained at a temperature T_0 and the lower boundary at a temperature T_1. The *boundary layer structure* and coordinate system are illustrated in Figure 6-39. The flow is divided into cellular *two-dimensional rolls* of width $\lambda/2$; alternate rolls rotate in opposite directions. The entire flow field is highly viscous. On the cold upper boundary a thin thermal boundary layer forms. When the two cold boundary layers from adjacent cells meet, they separate from the boundary and form a cold descending *thermal plume*. Similarly, a hot thermal boundary layer forms on the lower boundary of