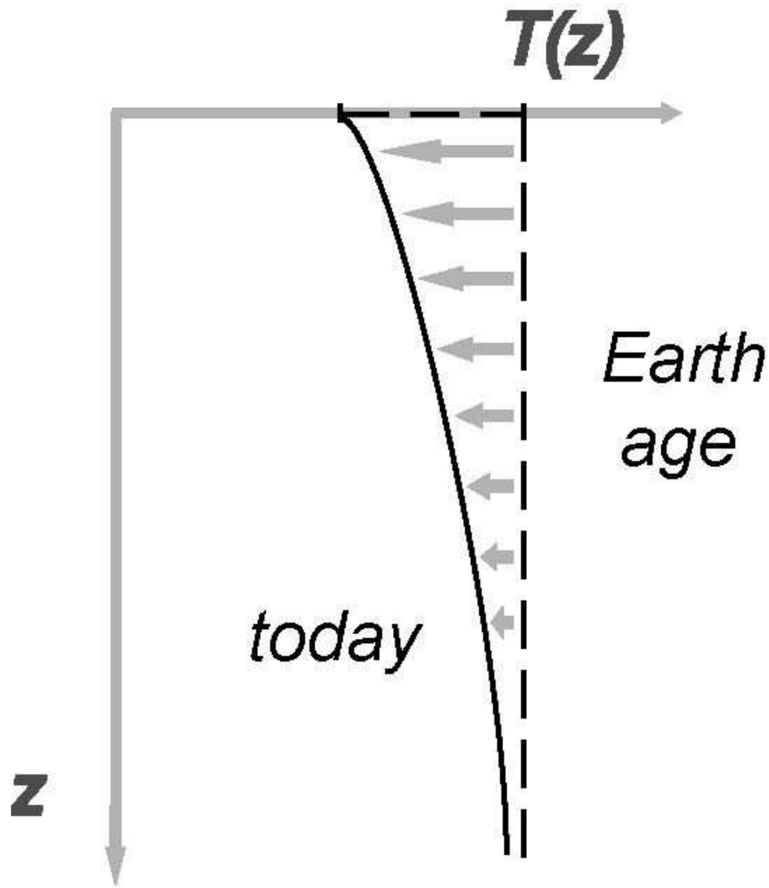
1. Cooling of an Isothermal Earth

Determinations of the temperature distribution within the Earth have long been a major focus of the physical sciences.
Early in the nineteenth century it was recognized from temperature measurements in mines, that the temperature T increased with depth z at a rate

$$\frac{dT}{dz} \sim 20 - 30\, K\, \text{km}^{-1}$$

the geothermal gradient. At this time, the heat flow at the Earth's surface implied by the geothermal gradient was attributed to the secular cooling of the planet, an inference that, as it turns out later, was only partially correct.

William Thompson (alias *Lord Kelvin*) used this assumption as the basis for his estimate of the age of the Earth. He assumed that the Earth was conductively cooling from an initial hot stage:



The distribution of temperature T at shallow depths under his assumptions can be modeled as one-dimensional, time-dependent heat conduction in the absence of heat sources as follows:

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} \tag{1}$$

In this heat conduction equation, ρ is the density, c_p is the specific heat, and k is the thermal conductivity.

The Earth surface is considered as a semi-infinite half-space defined by $z > 0$, which is initially at a temperature T_1. Upon planet formation, its surface temperature is suddenly decreased to temperature T_0 (at $t = 0$). This surface temperature T_0 is afterwards held constant for $t > 0$ until today. The solution to this problem, which also serves as the basic thermal model for the oceanic lithosphere, can be found in standard textbooks (e.g. Turcotte & Schubert, 1982) as

$$\frac{T_1 - T(z)}{T_1 - T_0} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{z}{2\sqrt{\kappa t}}} \exp\left(-\eta^2\right) d\eta \quad , \tag{2}$$

where $\kappa = k/\rho c_p$ is the thermal diffusivity (with units m^2/s).

Equation (2) cannot be solved analytically, but it is easy to see two simple limiting cases:

- At the surface ($z = 0$), the integral is zero and $T = T_0$.
- In the deep interior ($z \rightarrow \infty$), the integral is $\sqrt{\pi}/2$ and temperatures approach the limit T_1. .

Regions in the Earth where heat diffusion is an important heat transfer mechanism are usually referred to as thermal boundary layers.

We may define the thickness of this boundary layer, the so-called "thermal lithosphere", as the depth z_L where the left hand side of equation (2) is equal to 10%. From equation (2), it would follow for this special choice

$$z_L \sim 2.32\sqrt{\kappa t} \quad .$$

The outward heat flux q_0 at the lithosphere surface is given by differentiating equation (2) and evaluating the result at $z = 0$:

$$q_0 = k \left(\frac{\partial T}{\partial z}\right)_{z=0} = \frac{k\,(T_1 - T_0)}{\sqrt{\pi \kappa t}} \tag{3}$$

Equation (3) shows that the surface heat flux is proportional to the product of conductivity k and the temperature difference $(T_1 - T_0)$ and inversely proportional to the thermal boundary layer thickness $(\sqrt{\kappa t})$.

On the basis of this equation, Thompson proposed that the age of the Earth t_0 is given by

$$t_0 = \frac{(T_1 - T_0)^2}{\pi\,\kappa\,(\partial T/\partial z)^2_{z=0}} \quad , \tag{4}$$

where $(\partial T/\partial z)_{z=0}$ is the present geothermal gradient.

With $(\partial T/\partial z)_{z=0} = 25$ K km^{-1}, $T_1 - T_0 = 2000$ K, and κ = 1 mm^{2}s^{-1}, the age of the Earth from equation (4) is $t_0 \sim$ 65 Myr.

Lord Kelvin arrived at this age using the geothermal gradient measured in mines. The values of the temperature difference and the thermal diffusivity used were both reasonable. We now recognize, however, that the continental crust has a near-steady-state heat balance due to the heat generated by the heat-producing isotopes within the crust and the mantle heat flux from below (e.g. Sandiford, Tectonophysics 305 (1999) 109-120). His biggest historical mistake was, however, to neglect the main heat transport in the Earth mantle, i.e., thermal convection.

We want to use here the Lord Kelvin problem to demonstrate the availability of numerical methods to solve the 1D heat conduction equation (1) numerically.

2. Numerical solution

We want to solve equation (1) using the finite difference (FD) method. The solution of partial differential equations (PDEs) by means of FD is based on approximating derivatives of continuous functions by a discretized version of the derivative based on a discretized function. FD approximations can be derived through the use of Taylor series expansions.

The first step in the FD method is to construct a grid with points on which we are interested in solving the equation (this is called *discretization*).

The next step is to replace the continuous derivatives of eq. (1) with their finite difference counterparts. The time derivative $\partial T/\partial t$ can be approximated within a forward FD approximation as

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{n+1} - T_i^n}{t^{n+1} - t^n} = \frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{T_i^{new} - T_i^{current}}{\Delta t} \quad , \tag{5}$$

here T^n represents the temperature at the current time step, whereas T^{n+1} represents the new (future) temperature. The subscript i refers to the location (see figure). Both n and i are integers; n varies from 1 to n_t (total number of time steps) and i varies from 1 to n_x (total number of grid points in z-direction).

The spatial derivative is replaced by a central FD approximation, i.e.,

$$\frac{\partial^2 T}{\partial z^2} = \frac{\partial}{\partial z}\Big(\frac{\partial T}{\partial z}\Big) \approx \frac{\frac{T_{i+1}^n - T_i^n}{h} - \frac{T_i^n - T_{i-1}^n}{h}}{h} = \frac{T_{i+1}^n - 2\,T_i^n + T_{i-1}^n}{h^2} \tag{6}$$

Substituting equations (5) and (6) into equation (1) yield the so-called FTCS approximation (forward time centered space) for the considered parabolic PDE

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \kappa\Big(\frac{T_{i+1}^n - 2\,T_i^n + T_{i-1}^n}{h^2}\Big) \quad . \tag{7}$$

The third step is a rearrangement of the discretized equation, so that all known quantities (i.e. temperatures at time n) are on the right-hand side and the unknown quantities on the left-hand side (properties at $n+1$). This results in

$$T_i^{n+1} = T_i^n + \kappa\,\Delta t \left(\frac{T_{i+1}^n - 2\,T_i^n + T_{i-1}^n}{h^2}\right) \quad . \tag{8}$$

Because the temperature at the current timestep T^n is known, we can use eq. (8) to compute the new temperature without solving any additional equations. Such a scheme is called *explicit* and was made possible by the choice to evaluate the temporal derivative with forward differences.

The last step is to specify the initial and the boundary conditions. If the surface temperature T_0 is held constant at 300 K and the mantle temperature far away from the surface T_1 has the (asymptotic) value 2300 K, we can write the boundary conditions as follows

$$T(z = 0, t) = 300 \tag{9}$$

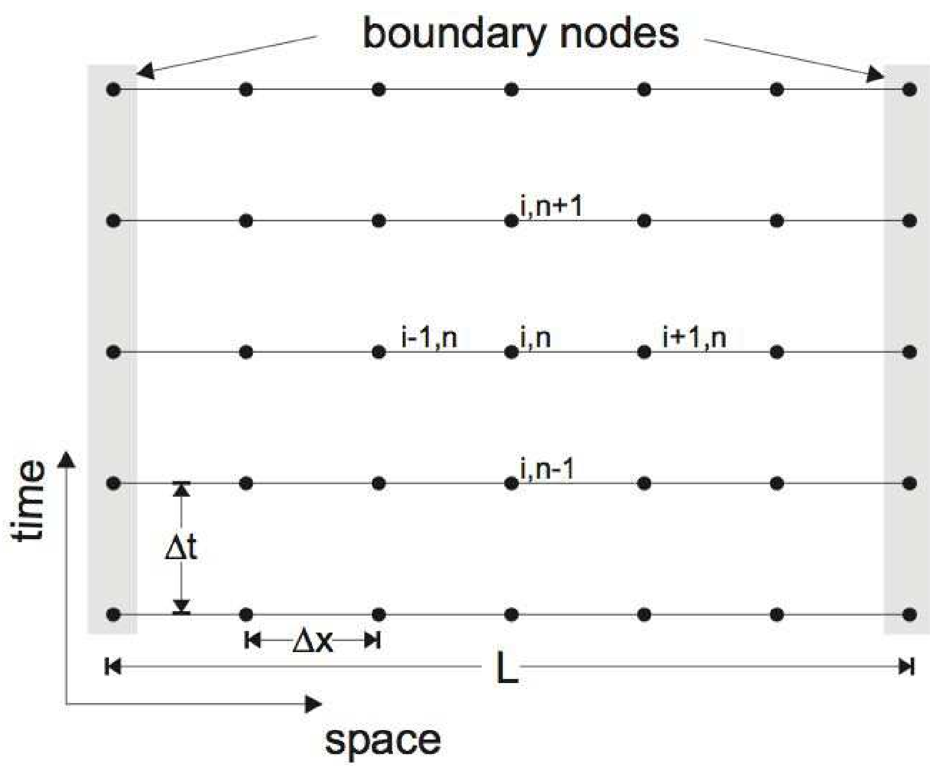
$$T(z = L, t) = 2300 \tag{10}$$

The initial step-like temperature profile is given simply with

$$T(z \leq 0, t = 0) = 300 \tag{11}$$

$$T(z > 0, t = 0) = 2300 \tag{12}$$

The attached MATLAB code shows an example in which the grid is initialized, and a time loop is performed. In the exercise, you will fill in the question marks and obtain a working code that solves equation (8).



Finite difference discretization of the 1D heat equation.
The finite difference method approximates the temperature at given grid points, with spacing h. The time evolution is computed at given times with timestep Δt.

3. Exercise

1. Open MATLAB and an editor and type the Matlab script in an empty file; alternatively use the template provided if you need inspiration.

   Save the file under the name heat1D.m. If starting from the template, fill in the question marks and then run the file by typing heat1D in the MATLAB command windows (make sure you are in the correct directory), or, alternatively, click RUN within the editor.

2. Study the time evolution of the spatial solution. Comment on the nature of the solution. What parameter determines the relationship between two spatial solutions at different times ?

   Does the temperature of the deep mantle matter for the nature of the solution ?

3. Vary the parameters (e.g. use more grid points, a larger or smaller timestep). Note that if the timestep is increased beyond a certain

value, the numerical method becomes unstable. What does this value depend on ? - This turns out as a major drawback of explicit finite difference codes such as the one presented here.

4. Record and plot the temperature evolution $T(z,t)$ versus time at a depth $z = 10$ km. Do the same with the thermal gradient $(\partial T/\partial z)_{z=0}$ at the surface. Compare the obtained numerical solution with the analytical expression according to eq. (3).

5. Think about how one would write a non-dimensionalized version of the temperature solver.

6. *Bonus question:* Derive a finite-difference approximation for the case of a variable thermal conductivity k. Test and compare the solutions for different depth dependencies of k. – A regarding recent study can be found, e.g., at McKenzie, Jackson & Priestley, EPSL 233 (2005) 337-349. [see `http://www.dynamicearth.de/compgeo/Tutorial/Day1/McKenzie2005.pdf`]